

A Geometric Algebra Perspective On Quantum Computational Gates And Universality In Quantum Computing

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We investigate the utility of geometric (Clifford) algebras (GA) methods in two specific applications to quantum information science. First, using the multiparticle spacetime algebra (MSTA, the geometric algebra of a relativistic configuration space), we present an *explicit* algebraic description of one and two-qubit quantum states together with a MSTA characterization of one and two-qubit quantum computational gates. Second, using the above mentioned characterization and the GA description of the Lie algebras $SO(3)$ and $SU(2)$ based on the rotor group $Spin^+(3, 0)$ formalism, we reexamine Boykin's proof of universality of quantum gates. We conclude that the MSTA approach does lead to a useful conceptual unification where the *complex* qubit space and the complex space of unitary operators acting on them become united, with both being made just by multivectors in *real* space. Finally, the GA approach to rotations based on the rotor group does bring conceptual and computational advantages compared to standard vectorial and matricial approaches.

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I. INTRODUCTION

Geometric (Clifford) algebra (GA) [1, 2] is a universal language for physics based on the mathematics of Clifford algebra. Applications of GA in physics span from quantum theory and gravity [3, 4] to classical electrodynamics [5] reaching even massive classical electrodynamics with Dirac's magnetic monopoles [6, 7].

There are some natural ways of including Clifford algebra and GA in quantum information science (QIS) motivated by physical reasons [8, 9]. For instance, any qubit (quantum bit, elementary carrier of quantum information) modelled as a spin- $\frac{1}{2}$ system can be regarded as a 2×2 matrix with an empty second column. All 2×2 matrices are combinations of Pauli matrices which are a representation of some GA. All 2×2 unitary transformations can be parametrized by elements of GA as well. Driven by such motivations, the first *formal* reformulations of some of the most important operations of quantum computing in the multiparticle geometric algebra formalism have been presented [10]. In a less conventional and very recent GA approach to quantum computing, the possibility of performing quantum-like algorithms using GA structures without involving quantum mechanics has been explored [11–13]. Within this approach, the standard tensor product is replaced by the geometric product and entangled states are replaced by multivectors with a geometrical interpretation in terms of "bags of shapes". Such GA approach brings new conceptual elements in QIS and the formalism of quantum computation loses its microscopic flavor when viewed from this novel GA point of view. Indeed, non-microworld implementations of quantum computing are suggested since there is no fundamental reason to believe that quantum computation has to be associated with physical systems described by quantum mechanics [13].

In [9], an extended discussion on applications of GA techniques in quantum information is presented, however the fundamental concept of universality in quantum computing is not discussed. In [10], the GA formulation of the Toffoli and Fredkin three-qubit quantum gates is introduced but no explicit characterization of all one and two-qubit quantum gates appears. Here, inspired by these two works, we present a (complementary and self-contained) compact, explicit and expository GA characterization of one and two qubit quantum gates together with a novel GA-based perspective on the concept of universality in quantum computation. First, we present an *explicit* multiparticle spacetime algebra (MSTA, the geometric Clifford algebra of a relativistic configuration space) [14–17] algebraic description of one and two-qubit quantum states (for instance, the 2-qubit Bell states) together with a MSTA characterization of one (bit-flip, phase-flip, combined bit and phase flip quantum gates, Hadamard gate, rotation gate, phase gate and $\frac{\pi}{8}$ -gate) and two-qubit quantum computational gates (CNOT, controlled-phase and SWAP quantum gates) [18]. Second, using the above mentioned explicit characterization and the GA description of the Lie algebras $SO(3)$ and $SU(2)$ based on the rotor group $Spin^+(3, 0)$ formalism, we reexamine Boykin's proof of universality of quantum gates [19, 20]. We conclude that the MSTA approach does lead to a useful conceptual unification where the complex qubit space and

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the complex space of unitary operators acting on them become united, with both being made just by multivectors in real space. Furthermore, the GA approach to rotations based on the rotor group clearly brings conceptual and computational advantages compared to standard vectorial and matricial approaches.

The layout of this article is as follows. In Section II, the basic MSTA formalism for the GA characterization of elementary gates for quantum computation is presented. In Section III, we present an explicit GA characterization of one and two-qubit quantum states together with a GA characterization of one and two-qubit quantum computational gates. Furthermore, we briefly discuss the extension of the MSTA formalism to density matrices. In Section IV, using the above mentioned explicit characterization and the GA description of the Lie algebras $SO(3)$ and $SU(2)$ based on the rotor group $Spin^+(3, 0)$ formalism, we reexamine Boykin's proof of universality of quantum gates [19, 20]. Finally, our final remarks are presented in Section V.

II. MULTIPARTICLE SPACETIME ALGEBRA

In this Section, we present the basic MSTA formalism for the GA characterization of elementary gates for quantum computation.

A. The n -Qubit Spacetime Algebra Formalism

It is commonly believed that complex space notions and an imaginary unit $i_{\mathbb{C}}$ are fundamental in quantum mechanics. However using spacetime algebra (STA, the geometric Clifford algebra of real 4-dimensional Minkowski spacetime, [2]) it has been shown how the $i_{\mathbb{C}}$ appearing in the Dirac, Pauli and Schrodinger equations has a geometrical interpretation in terms of rotations in real spacetime [21]. This becomes clear once introduced the geometric algebra of a relativistic configuration space, the so-called multiparticle spacetime algebra (MSTA) [14–17].

In the orthodox formulation of quantum mechanics, the tensor product is used in constructing both multiparticle states and many of the operators acting on these states. It is a notational device for explicitly isolating the Hilbert spaces of different particles. Geometric algebra attempts to justify from a foundational point of view the use of the tensor product in nonrelativistic quantum mechanics in terms of the underlying geometry of space-time [15]. The GA formalism provides an alternative representation of the tensor product in terms of the *geometric product*. Motivated by the usefulness of the STA formalism in describing a single-particle quantum mechanics, the MSTA approach to multiparticle quantum mechanics in both non-relativistic and relativistic settings was originally [15] introduced with the hope that it would also provide both computational and, above all, interpretational advances in multiparticle quantum theory. Conceptual advances are expected to arise by exploiting the special geometric insights that the MSTA approach provides. The unique feature of the MSTA is that it implies a separate copy of the time dimension for each particle, as well as the three spatial dimensions. It constitutes an attempt to construct a solid conceptual framework for a multi-time approach to quantum theory. Therefore, the main original motivation for using such formalism is the possibility of shedding light on issues of *locality* and *causality* in quantum theory. Indeed, interesting applications of the MSTA method devoted to the reexamination of Holland's causal interpretation of a system of two spin- $\frac{1}{2}$ particles [22] appear in [16, 17]. Following this line of investigation, in this article we apply the MSTA method to express qubits and quantum gates and to revisit in geometric algebra terms Boykin's proof of universality in quantum computing.

The multiparticle spacetime algebra provides the ideal algebraic structure for studying multiparticle states and operators. MSTA is the geometric algebra of n -particle configuration space which, for relativistic systems, consists of n copies (each copy is a 1-particle space) of Minkowski spacetime. A suitable basis for the MSTA is given by the set $\{\gamma_{\mu}^a\}$, where $\mu = 0, \dots, 3$ labels the spacetime vector and $a = 1, \dots, n$ labels the particle space. These basis vectors satisfy the orthogonality conditions $\gamma_{\mu}^a \cdot \gamma_{\nu}^b = \delta^{ab} \eta_{\mu\nu}$ with $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. Vectors from different particle spaces anticommute as a consequence of their orthogonality. Note that a basis for the entire MSTA has 2^{4n} degrees of freedom, $\dim_{\mathbb{R}}[\mathfrak{cl}(1, 3)]^n = 2^{4n}$. In nonrelativistic quantum mechanics, all of the individual time coordinates are identified with a single absolute time. We take this vector to be γ_0^a for each a . Spatial vectors relative to these timelike vectors are modeled as bivectors through a spacetime split. A basis set of relative vectors is then defined by $\sigma_k^a \stackrel{\text{def}}{=} \gamma_k^a \gamma_0^a$, with $k = 1, \dots, 3$ and $a = 1, \dots, n$. For each particle space the set $\{\sigma_k^a\}$ generates the geometric algebra of relative space $\mathfrak{cl}(3) \cong \mathfrak{cl}^+(1, 3)$. Each particle space has a basis given by,

$$1, \{\sigma_k\}, \{i\sigma_k\}, i, \quad (1)$$

where the volume element i (the *pseudoscalar*, the highest grade multivector) is defined by $i \stackrel{\text{def}}{=} \sigma_1 \sigma_2 \sigma_3$ (suppressing the particle space indices). The basis in (1) defines the Pauli algebra (the geometric algebra of the 3-dimensional

Euclidean space, [2]) but in GA the three Pauli σ_k are no longer viewed as three matrix-valued components of a single isospace vector, but as three independent basis vectors for real space. Notice that unlike spacetime basis vectors, relative vectors $\{\sigma_k^a\}$ from separate particle spaces commute, $\sigma_k^a \sigma_j^b = \sigma_j^b \sigma_k^a$, $a \neq b$. It turns out that the $\{\sigma_k^a\}$ generate the direct product space $[\mathbf{cl}(3)]^n \stackrel{\text{def}}{=} \mathbf{cl}(3) \otimes \dots \otimes \mathbf{cl}(3)$ of n copies of the geometric algebra of the 3-dimensional Euclidean space. Within the MSTA formalism, Pauli spinors (a spin- $\frac{1}{2}$ quantum system is an adequate model of quantum bit) may be represented as elements of the even subalgebra of the Pauli algebra spanned by $\{1, i\sigma_k\}$ and isomorphic to the quaternion algebra. This space is a 4-dimensional real space where a general even element can be written as, $\psi = a^0 + a^k i\sigma_k$, where a^0 and a^k with $k = 1, 2, 3$ are *real* scalars. An ordinary quantum state contains a pair of *complex* numbers, α and β ,

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \text{Re } \alpha + i_{\mathbb{C}} \text{Im } \alpha \\ \text{Re } \beta + i_{\mathbb{C}} \text{Im } \beta \end{pmatrix}. \quad (2)$$

In [14], it was established a $1 \leftrightarrow 1$ map between Pauli column spinors and elements of the even subalgebra,

$$|\psi\rangle = \begin{pmatrix} a^0 + i_{\mathbb{C}} a^3 \\ -a^2 + i_{\mathbb{C}} a^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k i\sigma_k. \quad (3)$$

where the coefficients a^0 and $a^k \in \mathbb{R}$. Multivectors $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ are the computational basis states of the real 4-dimensional even subalgebra corresponding to the two-dimensional Hilbert space \mathcal{H}_2^1 with standard computational basis given by $\mathcal{B}_{\mathcal{H}_2^1} \stackrel{\text{def}}{=} \{|0\rangle, |1\rangle\}$. In the GA formalism,

$$|0\rangle \leftrightarrow \psi_{|0\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} 1, \quad |1\rangle \leftrightarrow \psi_{|1\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} -i\sigma_2. \quad (4)$$

The action of the conventional quantum Pauli operators $\{\hat{\Sigma}_k, i_{\mathbb{C}} \hat{I}\}$ translates as [14],

$$\hat{\Sigma}_k |\psi\rangle \leftrightarrow \sigma_k \psi \sigma_3, \quad (5)$$

with $k = 1, 2, 3$ and,

$$i_{\mathbb{C}} |\psi\rangle \leftrightarrow \psi i\sigma_3. \quad (6)$$

In synthesis, in the single-particle theory, non-relativistic states are constructed from the even subalgebra of the Pauli algebra with a basis provided by the set $\{1, i\sigma_k\}$ with $k = 1, 2, 3$. The role of the (single) imaginary unit of conventional quantum theory is played by right multiplication by $i\sigma_3$. Verifying that this translation scheme works properly is just a matter of simple computations. Indeed, from (3) and (5) we obtain,

$$\begin{aligned} \hat{\Sigma}_1 |\psi\rangle &= \begin{pmatrix} -a^2 + i_{\mathbb{C}} a^1 \\ a^0 + i_{\mathbb{C}} a^3 \end{pmatrix} \leftrightarrow -a^2 + a^3 i\sigma_1 - a^0 i\sigma_2 + a^1 i\sigma_3 = \sigma_1 (a^0 + a^k i\sigma_k) \sigma_3, \\ \hat{\Sigma}_2 |\psi\rangle &= \begin{pmatrix} a^1 + i_{\mathbb{C}} a^2 \\ -a^3 + i_{\mathbb{C}} a^0 \end{pmatrix} \leftrightarrow a^1 + a^0 i\sigma_1 + a^3 i\sigma_2 + a^2 i\sigma_3 = \sigma_2 (a^0 + a^k i\sigma_k) \sigma_3, \\ \hat{\Sigma}_3 |\psi\rangle &= \begin{pmatrix} a^0 + i_{\mathbb{C}} a^3 \\ a^2 - i_{\mathbb{C}} a^1 \end{pmatrix} \leftrightarrow a^0 - a^1 i\sigma_1 - a^2 i\sigma_2 + a^3 i\sigma_3 = \sigma_3 (a^0 + a^k i\sigma_k) \sigma_3. \end{aligned} \quad (7)$$

In the n -particle algebra there will be n -copies of $i\sigma_3$, namely $i\sigma_3^a$ with $a = 1, \dots, n$. However, in order to faithfully mirror conventional quantum mechanics, the right-multiplication by all of these must yield the same result. Therefore, it must be

$$\psi i\sigma_3^1 = \psi i\sigma_3^2 = \dots = \psi i\sigma_3^{n-1} = \psi i\sigma_3^n. \quad (8)$$

Relations in (8) are obtained by introducing the n -particle correlator E_n defined as,

$$E_n \stackrel{\text{def}}{=} \prod_{b=2}^n \frac{1}{2} (1 - i\sigma_3^1 i\sigma_3^b), \quad (9)$$

and satisfying $E_n i\sigma_3^a = E_n i\sigma_3^b = J_n$; $\forall a, b$. Notice that E_n in (9) has been defined by picking out the $a = 1$ space and correlating all the other spaces to this. However, the value of E_n is independent of which of the n spaces is chosen and

correlated to. The complex structure is defined by $J_n = E_n i\sigma_3^a$ with $J_n^2 = -E_n$. Right-multiplication by the quantum correlator E_n is a projection operation that reduces the number of *real* degrees of freedom from $4^n = \dim_{\mathbb{R}} [\mathfrak{cl}^+(3)]^n$ to the expected $2^{n+1} = \dim_{\mathbb{R}} \mathcal{H}_2^n$. The projection can be interpreted physically as locking the phases of the various particles together. The *reduced* even subalgebra space will be denoted by $[\mathfrak{cl}^+(3)]^n / E_n$. Multivectors belonging to this space can be regarded as n -particle spinors (or, n -qubit states), analogous to $\mathfrak{cl}^+(3)$ for a single particle. In synthesis, the extension to multiparticle systems involves a separate copy of the STA for each particle and the standard imaginary unit induces correlations between these particle spaces.

B. An Example: The 2-Qubit Spacetime Algebra Formalism

Quantum theory works with a single imaginary unit $i_{\mathbb{C}}$, but in the 2-particle algebra there are two bivectors playing the role of $i_{\mathbb{C}}$, $i\sigma_3^1$ and $i\sigma_3^2$. Right-multiplication of a state by either of these has to result in the same state in order for the GA treatment to faithfully mirror standard quantum mechanics. Therefore it must be,

$$\psi i\sigma_3^1 = \psi i\sigma_3^2. \quad (10)$$

Manipulation of (10) yields $\psi = \psi E$, where,

$$E \stackrel{\text{def}}{=} \frac{1}{2} (1 - i\sigma_3^1 i\sigma_3^2), \quad E^2 = E. \quad (11)$$

Right-multiplication by E is a projection operation. If we include this factor on the right of all states, the number of *real* degrees of freedom decrease from 16 to the expected 8. The multivectorial basis $\mathcal{B}_{\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)}$ spanning the 16-dimensional geometric algebra $\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)$ is given by,

$$\mathcal{B}_{\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)} \stackrel{\text{def}}{=} \{1, i\sigma_l^1, i\sigma_k^2, i\sigma_l^1 i\sigma_k^2\}, \quad (12)$$

with k and $l = 1, 2, 3$. Right-multiplying the multivectors in $\mathcal{B}_{\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)}$ by the quantum projection operator E , we obtain,

$$\mathcal{B}_{\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)} \xrightarrow{E} \mathcal{B}_{\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)} E \stackrel{\text{def}}{=} \{E, i\sigma_l^1 E, i\sigma_k^2 E, i\sigma_l^1 i\sigma_k^2 E\}. \quad (13)$$

After some straightforward algebra, it follows that,

$$\begin{aligned} E &= -i\sigma_3^1 i\sigma_3^2 E, \quad i\sigma_1^2 E = -i\sigma_3^1 i\sigma_2^2 E, \quad i\sigma_2^2 E = i\sigma_3^1 i\sigma_1^2 E, \quad i\sigma_3^2 E = i\sigma_3^1 E, \\ i\sigma_1^1 E &= -i\sigma_2^1 i\sigma_3^2 E, \quad i\sigma_1^1 i\sigma_1^2 E = -i\sigma_2^1 i\sigma_2^2 E, \quad i\sigma_1^1 i\sigma_2^2 E = i\sigma_2^1 i\sigma_1^2 E, \quad i\sigma_1^1 i\sigma_3^2 E = i\sigma_2^1 E. \end{aligned} \quad (14)$$

Therefore, a suitable basis for the 8-dimensional *reduced* even subalgebra $[\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)] / E$ is given by,

$$\mathcal{B}_{[\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)] / E} \stackrel{\text{def}}{=} \{1, i\sigma_1^2, i\sigma_2^2, i\sigma_3^2, i\sigma_1^1, i\sigma_1^1 i\sigma_1^2, i\sigma_1^1 i\sigma_2^2, i\sigma_1^1 i\sigma_3^2\}. \quad (15)$$

The basis in (15) spans $[\mathfrak{cl}^+(3) \otimes \mathfrak{cl}^+(3)] / E$ and is the analog of a suitable standard complex basis spanning the complex Hilbert space \mathcal{H}_2^2 . The spacetime algebra representation of a direct-product 2-particle Pauli spinor (a 2-qubits quantum state) is now given by $\psi^1 \phi^2 E$, where ψ^1 and ϕ^2 are spinors (even multivectors) in their own spaces, $|\psi, \phi\rangle \leftrightarrow \psi^1 \phi^2 E$. A GA version of a standard complete basis for 2-particle spin states is provided by,

$$\begin{aligned} |0\rangle \otimes |0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow E, \quad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow -i\sigma_2^2 E, \\ |1\rangle \otimes |0\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow -i\sigma_2^1 E, \quad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow i\sigma_2^1 i\sigma_2^2 E. \end{aligned} \quad (16)$$

For instance, a standard entangled state between a pair of 2-level systems, a spin singlet state is defined as,

$$|\psi_{\text{singlet}}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \quad (17)$$

From (11), (16) and (17), it follows that the GA analog of $|\psi_{\text{singlet}}\rangle$ is given by,

$$\mathcal{H}_2^2 \ni |\psi_{\text{singlet}}\rangle \leftrightarrow \psi_{\text{singlet}}^{(\text{GA})} \in [\mathfrak{cl}^+(3)]^2, \quad (18)$$

where,

$$\psi_{\text{singlet}}^{(\text{GA})} = \frac{1}{2^{\frac{3}{2}}} (i\sigma_2^1 - i\sigma_2^2) (1 - i\sigma_3^1 i\sigma_3^2). \quad (19)$$

Furthermore, the role of multiplication by the quantum imaginary $i_{\mathbb{C}}$ for 2-particle states is taken by right-sided multiplication by J ,

$$J = Ei\sigma_3^1 = Ei\sigma_3^2 = \frac{1}{2} (i\sigma_3^1 + i\sigma_3^2), \quad (20)$$

so that $J^2 = -E$. The action of 2-particle Pauli operators is the following,

$$\hat{\Sigma}_k \otimes \hat{I} |\psi\rangle \leftrightarrow -i\sigma_k^1 \psi J, \quad \hat{\Sigma}_k \otimes \hat{\Sigma}_l |\psi\rangle \leftrightarrow -i\sigma_k^1 i\sigma_l^2 \psi E, \quad \hat{I} \otimes \hat{\Sigma}_k |\psi\rangle \leftrightarrow -i\sigma_k^2 \psi J. \quad (21)$$

For instance, the second Equation in (21) follows from the following line of reasoning,

$$\hat{\Sigma}_l^2 |\psi\rangle \leftrightarrow \sigma_l^2 \psi \sigma_3^2 = \sigma_l^2 \psi E \sigma_3^2 = -\sigma_l^2 \psi E i i \sigma_3^2 = -i\sigma_l^2 \psi E i \sigma_3^2 = -i\sigma_l^2 \psi J, \quad (22)$$

and therefore,

$$\hat{\Sigma}_k \otimes \hat{\Sigma}_l |\psi\rangle \leftrightarrow (-i\sigma_k^1) (-i\sigma_l^2) \psi J^2 = -i\sigma_k^1 i\sigma_l^2 \psi E. \quad (23)$$

Finally, recalling that $i_{\mathbb{C}} \hat{\Sigma}_k |\psi\rangle \leftrightarrow i\sigma_k \psi$, we point out that,

$$i_{\mathbb{C}} \hat{\Sigma}_k \otimes \hat{I} |\psi\rangle \leftrightarrow i\sigma_k^1 \psi \text{ and } \hat{I} \otimes i_{\mathbb{C}} \hat{\Sigma}_k |\psi\rangle \leftrightarrow i\sigma_k^2 \psi. \quad (24)$$

More details on the MSTA formalism can be found in [14–17].

III. GEOMETRIC ALGEBRA AND QUANTUM COMPUTATION

Interesting quantum computations may require constructions of complicated computational networks with several gates acting on n -qubits defining a non-trivial quantum algorithm. Therefore it is of great practical importance finding a convenient *universal* set of quantum gates. A set of quantum gates $\{\hat{U}_i\}$ is said to be *universal* if any logical operation \hat{U}_L can be written as [18],

$$\hat{U}_L = \prod_{\hat{U}_i \in \{\hat{U}_i\}} \hat{U}_i. \quad (25)$$

In this Section, we present an explicit GA characterization of 1 and 2-qubit quantum states together with a GA characterization of a universal set of quantum gates for quantum computation. Furthermore, we mention the extension of the MSTA formalism to density matrices.

A. Geometric Algebra and 1-Qubit Quantum Computing

We consider simple circuit models of quantum computation with 1-qubit quantum gates in the GA formalism.

Quantum NOT Gate (or Bit Flip Quantum Gate). A nontrivial reversible operation we can apply to a single qubit is the NOT operation (gate) denoted by the symbol $\hat{\Sigma}_1$. For the sake of simplicity, we will first study the action of quantum gates in the GA formalism acting on 1-qubit quantum gates given by $\psi_{|q\rangle}^{(\text{GA})} = a^0 + a^2 i\sigma_2$. Then, $\hat{\Sigma}_1^{(\text{GA})}$ is defined as,

$$\hat{\Sigma}_1 |q\rangle \stackrel{\text{def}}{=} |q \oplus 1\rangle \leftrightarrow \psi_{|q \oplus 1\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} \sigma_1 (a^0 + a^2 i\sigma_2) \sigma_3. \quad (26)$$

Recalling that the unit pseudoscalar $i \stackrel{\text{def}}{=} \sigma_1 \sigma_2 \sigma_3$ is such that $i \sigma_k = \sigma_k i$ with $k = 1, 2, 3$ and recalling the geometric product rule,

$$\sigma_i \sigma_j = \sigma_i \cdot \sigma_j + \sigma_i \wedge \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k, \quad (27)$$

we obtain,

$$\hat{\Sigma}_1 |q\rangle \stackrel{\text{def}}{=} |q \oplus 1\rangle \leftrightarrow \psi_{|q \oplus 1\rangle}^{(\text{GA})} = -(a^2 + a^0 i \sigma_2). \quad (28)$$

For the sake of completeness, we point out that the unitary quantum gate $\hat{\Sigma}_1^{(\text{GA})}$ acts on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ as follows,

$$\hat{\Sigma}_1^{(\text{GA})} : 1 \rightarrow -i\sigma_2, \hat{\Sigma}_1^{(\text{GA})} : i\sigma_1 \rightarrow i\sigma_3, \hat{\Sigma}_1^{(\text{GA})} : i\sigma_2 \rightarrow -1, \hat{\Sigma}_1^{(\text{GA})} : i\sigma_3 \rightarrow i\sigma_1. \quad (29)$$

Phase Flip Quantum Gate. Another nontrivial reversible operation we can apply to a single qubit is the phase flip gate denoted by the symbol $\hat{\Sigma}_3$. In the GA formalism, the action of the unitary quantum gate $\hat{\Sigma}_3^{(\text{GA})}$ on the multivector $\psi_{|q\rangle}^{(\text{GA})} = a^0 + a^2 i \sigma_2$ is given by,

$$\hat{\Sigma}_3 |q\rangle \stackrel{\text{def}}{=} (-1)^q |q\rangle \leftrightarrow \psi_{(-1)^q |q\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} \sigma_3 (a^0 + a^2 i \sigma_2) \sigma_3. \quad (30)$$

From (10) and (27) it turns out that,

$$\hat{\Sigma}_3 |q\rangle \stackrel{\text{def}}{=} (-1)^q |q\rangle \leftrightarrow \psi_{(-1)^q |q\rangle}^{(\text{GA})} = a^0 - a^2 i \sigma_2. \quad (31)$$

Finally, the unitary quantum gate $\hat{\Sigma}_3^{(\text{GA})}$ acts on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ in the following manner,

$$\hat{\Sigma}_3^{(\text{GA})} : 1 \rightarrow 1, \hat{\Sigma}_3^{(\text{GA})} : i\sigma_1 \rightarrow -i\sigma_1, \hat{\Sigma}_3^{(\text{GA})} : i\sigma_2 \rightarrow -i\sigma_2, \hat{\Sigma}_3^{(\text{GA})} : i\sigma_3 \rightarrow i\sigma_3. \quad (32)$$

Combined Bit and Phase Flip Quantum Gates. A suitable combination of the two reversible operations $\hat{\Sigma}_1$ and $\hat{\Sigma}_3$ gives rise to another nontrivial reversible operation we can apply to a single qubit. Such operation is denoted by the symbol $\hat{\Sigma}_2 \stackrel{\text{def}}{=} i_{\mathbb{C}} \hat{\Sigma}_1 \circ \hat{\Sigma}_3$. The action of $\hat{\Sigma}_2^{(\text{GA})}$ on $\psi_{|q\rangle}^{(\text{GA})} = a^0 + a^2 i \sigma_2$ is given by,

$$\hat{\Sigma}_2 |q\rangle \stackrel{\text{def}}{=} i_{\mathbb{C}} (-1)^q |q \oplus 1\rangle \leftrightarrow \psi_{i_{\mathbb{C}}(-1)^q |q \oplus 1\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} \sigma_2 (a^0 + a^2 i \sigma_2) \sigma_3. \quad (33)$$

From (10) and (27) it turns out that,

$$\hat{\Sigma}_2 |q\rangle \stackrel{\text{def}}{=} i_{\mathbb{C}} (-1)^q |q \oplus 1\rangle \leftrightarrow \psi_{i_{\mathbb{C}}(-1)^q |q \oplus 1\rangle}^{(\text{GA})} = (a^2 - a^0 i \sigma_2) i \sigma_3. \quad (34)$$

Indeed, using (27) and the fact that $i \sigma_k = \sigma_k i$ for $k = 1, 2, 3$, we obtain,

$$\sigma_2 (a^0 + a^2 i \sigma_2) \sigma_3 = (a^2 - a^0 i \sigma_2) i \sigma_3. \quad (35)$$

Finally, the action of the unitary quantum gate $\hat{\Sigma}_2^{(\text{GA})}$ on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ is,

$$\hat{\Sigma}_2^{(\text{GA})} : 1 \rightarrow i\sigma_1, \hat{\Sigma}_2^{(\text{GA})} : i\sigma_1 \rightarrow 1, \hat{\Sigma}_2^{(\text{GA})} : i\sigma_2 \rightarrow i\sigma_3, \hat{\Sigma}_2^{(\text{GA})} : i\sigma_3 \rightarrow i\sigma_2. \quad (36)$$

Hadamard Quantum Gate. The GA analog of the Walsh-Hadamard quantum gate $\hat{H} \stackrel{\text{def}}{=} \frac{\hat{\Sigma}_1 + \hat{\Sigma}_3}{\sqrt{2}}$, $\hat{H}^{(\text{GA})}$ acts on $\psi_{|q\rangle}^{(\text{GA})} = a^0 + a^2 i \sigma_2$ as follows,

$$\hat{H} |q\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} [|q \oplus 1\rangle + (-1)^q |q\rangle] \leftrightarrow \psi_{\hat{H}|q\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} \left(\frac{\sigma_1 + \sigma_3}{\sqrt{2}} \right) (a^0 + a^2 i \sigma_2) \sigma_3. \quad (37)$$

Using (28) and (31), (37) becomes,

$$\hat{H} |q\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} [|q \oplus 1\rangle + (-1)^q |q\rangle] \leftrightarrow \psi_{\hat{H}|q\rangle}^{(\text{GA})} = \frac{a^0}{\sqrt{2}} (1 - i \sigma_2) - \frac{a^2}{\sqrt{2}} (1 + i \sigma_2). \quad (38)$$

Notice that GA versions of the Hadamard transformed computational states, $|+\rangle$ and $|-\rangle$, are given by,

$$|+\rangle \stackrel{\text{def}}{=} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \leftrightarrow \psi_{|+\rangle}^{(\text{GA})} = \frac{1 - i\sigma_2}{\sqrt{2}} \text{ and, } |-\rangle \stackrel{\text{def}}{=} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \leftrightarrow \psi_{|-\rangle}^{(\text{GA})} = \frac{1 + i\sigma_2}{\sqrt{2}}, \quad (39)$$

respectively. Finally, the unitary quantum gate $\hat{H}^{(\text{GA})}$ acts on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ as follows,

$$\hat{H}^{(\text{GA})} : 1 \rightarrow \frac{1 - i\sigma_2}{\sqrt{2}}, \hat{H}^{(\text{GA})} : i\sigma_1 \rightarrow \frac{-i\sigma_1 + i\sigma_3}{\sqrt{2}}, \hat{H}^{(\text{GA})} : i\sigma_2 \rightarrow -\frac{1 + i\sigma_2}{\sqrt{2}}, \hat{H}^{(\text{GA})} : i\sigma_3 \rightarrow \frac{i\sigma_1 + i\sigma_3}{\sqrt{2}}. \quad (40)$$

Rotation Gate. The action of rotation gates $\hat{R}_\theta^{(\text{GA})}$ on $\psi_{|q\rangle}^{(\text{GA})} = a^0 + a^2 i\sigma_2$ is defined as,

$$\hat{R}_\theta |q\rangle \stackrel{\text{def}}{=} \left[\frac{1 + \exp(i\mathbb{C}\theta)}{2} + (-1)^q \frac{1 - \exp(i\mathbb{C}\theta)}{2} \right] |q\rangle \leftrightarrow \psi_{\hat{R}_\theta |q\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} a^0 + a^2 i\sigma_2 (\cos \theta + i\sigma_3 \sin \theta). \quad (41)$$

The unitary quantum gate $\hat{R}_\theta^{(\text{GA})}$ acts on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ as follows,

$$\hat{R}_\theta^{(\text{GA})} : 1 \rightarrow 1, \hat{R}_\theta^{(\text{GA})} : i\sigma_1 \rightarrow i\sigma_1 (\cos \theta + i\sigma_3 \sin \theta), \hat{R}_\theta^{(\text{GA})} : i\sigma_2 \rightarrow i\sigma_2 (\cos \theta + i\sigma_3 \sin \theta), \hat{R}_\theta^{(\text{GA})} : i\sigma_3 \rightarrow i\sigma_3. \quad (42)$$

Phase Quantum Gate and $\frac{\pi}{8}$ -Quantum Gate. The phase gate $\hat{S}^{(\text{GA})}$ acts on $\psi_{|q\rangle}^{(\text{GA})} = a^0 + a^2 i\sigma_2$ as follows,

$$\hat{S} |q\rangle \stackrel{\text{def}}{=} \left[\frac{1 + i\mathbb{C}}{2} + (-1)^q \frac{1 - i\mathbb{C}}{2} \right] |q\rangle \leftrightarrow \psi_{\hat{S} |q\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} a^0 + (a^2 i\sigma_2) i\sigma_3. \quad (43)$$

Furthermore, the unitary quantum gate $\hat{S}^{(\text{GA})}$ acts on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ as follows,

$$\hat{S}^{(\text{GA})} : 1 \rightarrow 1, \hat{S}^{(\text{GA})} : i\sigma_1 \rightarrow i\sigma_2, \hat{S}^{(\text{GA})} : i\sigma_2 \rightarrow -i\sigma_1, \hat{S}^{(\text{GA})} : i\sigma_3 \rightarrow i\sigma_3. \quad (44)$$

The GA analog of the $\frac{\pi}{8}$ -quantum gate \hat{T} is defined as,

$$\hat{T} |q\rangle \stackrel{\text{def}}{=} \left[\frac{1 + \exp(i\mathbb{C}\frac{\pi}{4})}{2} + (-1)^q \frac{1 - \exp(i\mathbb{C}\frac{\pi}{4})}{2} \right] |q\rangle \leftrightarrow \psi_{\hat{T} |q\rangle}^{(\text{GA})} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (a^0 + a^2 i\sigma_2) (1 + i\sigma_3). \quad (45)$$

Finally, the unitary quantum gate $\hat{T}^{(\text{GA})}$ acts on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ as follows,

$$\hat{T}^{(\text{GA})} : 1 \rightarrow 1, \hat{T}^{(\text{GA})} : i\sigma_1 \rightarrow i\sigma_1 \frac{(1 + i\sigma_3)}{\sqrt{2}}, \hat{T}^{(\text{GA})} : i\sigma_2 \rightarrow i\sigma_2 \frac{(1 + i\sigma_3)}{\sqrt{2}}, \hat{T}^{(\text{GA})} : i\sigma_3 \rightarrow i\sigma_3. \quad (46)$$

In conclusion, the action of some of the most relevant 1-qubit quantum gates in the GA formalism on the GA computational basis states $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ can be summarized in the following tabular form:

1-Qubit States	NOT	Phase Flip	Bit and Phase Flip	Hadamard	Rotation	$\frac{\pi}{8}$ -Gate
1	$-i\sigma_2$	1	$i\sigma_1$	$\frac{1-i\sigma_2}{\sqrt{2}}$	1	1
$i\sigma_1$	$i\sigma_3$	$-i\sigma_1$	1	$\frac{-i\sigma_1+i\sigma_3}{\sqrt{2}}$	$i\sigma_1 (\cos \theta + i\sigma_3 \sin \theta)$	$i\sigma_1 \frac{(1+i\sigma_3)}{\sqrt{2}}$
$i\sigma_2$	-1	$-i\sigma_2$	$i\sigma_3$	$-\frac{1+i\sigma_2}{\sqrt{2}}$	$i\sigma_2 (\cos \theta + i\sigma_3 \sin \theta)$	$i\sigma_2 \frac{(1+i\sigma_3)}{\sqrt{2}}$
$i\sigma_3$	$i\sigma_1$	$i\sigma_3$	$i\sigma_2$	$\frac{i\sigma_1+i\sigma_3}{\sqrt{2}}$	$i\sigma_3$	$i\sigma_3 \rightarrow i\sigma_3$

(47)

Therefore, in the GA approach qubits become elements of the even subalgebra, unitary quantum gates become rotors and the conventional complex structure of quantum mechanics is controlled by the bivector $i\sigma_3$.

Quantum gates have a geometrical interpretation when expressed in the GA formalism. Recall that in the conventional approach to quantum gates, an arbitrary unitary operator on a single qubit can be written as a combination of rotations together with global phase shifts on the qubit, $\hat{U} = e^{i\mathbb{C}\alpha} R_{\hat{n}}(\theta)$ for some *real* numbers α and θ and a *real* three-dimensional unit vector $\hat{n} \equiv (n_1, n_2, n_3)$. For instance, the Hadamard gate \hat{H} acting on a single qubit has the properties $\hat{H}\hat{\Sigma}_1\hat{H} = \hat{\Sigma}_3$ and $\hat{H}\hat{\Sigma}_3\hat{H} = \hat{\Sigma}_1$ with $\hat{H}^2 = \hat{I}$. Therefore, \hat{H} can be envisioned (up to an overall phase) as

a $\theta = \pi$ rotation about the axis $\hat{n} = \frac{1}{\sqrt{2}}(\hat{n}_1 + \hat{n}_3)$ that rotates \hat{x} to \hat{z} and viceversa, $\hat{H} = -i_{\mathbb{C}} R_{\frac{1}{\sqrt{2}}(\hat{n}_1 + \hat{n}_3)}(\pi)$. In GA, rotations are handled by means of rotors. The Hadamard gate, for instance, has a simple *real* (no use of *complex* numbers is needed) geometric interpretation: it is represented by a rotor $\hat{H}^{(\text{GA})} = e^{-i\frac{\pi}{2}\frac{\sigma_1 + \sigma_3}{\sqrt{2}}}$ describing a rotation by π about the $\frac{\sigma_1 + \sigma_3}{\sqrt{2}}$ axis. It is straightforward to show that the action of the rotor $\hat{H}^{(\text{GA})}$ on the 1-qubit computational basis states satisfies (up to an overall irrelevant phase shift) the transformation laws appearing in (47). We point out that when the Hadamard gate is represented by a rotor for a rotation by π , $\hat{H}^{(\text{GA})2} = -1$. Therefore, it seems that the gate is more accurately represented by a reflection rather than a rotation. The phase difference may be important when state amplitudes transformed by the Hadamard gate are combined with the ones transformed by other gates. In [9], it was also proposed treating the Hadamard gate as a rotation but it is now recognized the problem with this interpretation. Similar geometric considerations could be carried out for the other 1-qubit gates [9].

B. Geometric Algebra and 2-Qubit Quantum Computing

We consider simple circuit models of quantum computation with 2-qubit quantum gates in the GA formalism. Before doing so, we present an explicit MSTA description of quantum Bell states.

Geometric Algebra and Bell States. We present a GA characterization of the set of maximally entangled 2-qubits Bell states. Bell states are an important example of maximally entangled quantum states and form an orthonormal basis $\mathcal{B}_{\text{Bell}}$ in the product Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. Consider the 2-qubit computational basis $\mathcal{B}_{\text{computational}} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, then the four Bell states can be constructed as follows [18],

$$\begin{aligned} |0\rangle \otimes |0\rangle &\rightarrow |\psi_{\text{Bell}_1}\rangle \stackrel{\text{def}}{=} [\hat{U}_{\text{CNOT}} \circ (\hat{H} \otimes \hat{I})] (|0\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \\ |0\rangle \otimes |1\rangle &\rightarrow |\psi_{\text{Bell}_2}\rangle \stackrel{\text{def}}{=} [\hat{U}_{\text{CNOT}} \circ (\hat{H} \otimes \hat{I})] (|0\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \\ |1\rangle \otimes |0\rangle &\rightarrow |\psi_{\text{Bell}_3}\rangle \stackrel{\text{def}}{=} [\hat{U}_{\text{CNOT}} \circ (\hat{H} \otimes \hat{I})] (|1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\ |1\rangle \otimes |1\rangle &\rightarrow |\psi_{\text{Bell}_4}\rangle \stackrel{\text{def}}{=} [\hat{U}_{\text{CNOT}} \circ (\hat{H} \otimes \hat{I})] (|1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle). \end{aligned} \quad (48)$$

In (48), \hat{H} is the Hadamard gate and \hat{U}_{CNOT} is the CNOT gate. The Bell basis in $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ is given by,

$$\mathcal{B}_{\text{Bell}} \stackrel{\text{def}}{=} \{|\psi_{\text{Bell}_1}\rangle, |\psi_{\text{Bell}_2}\rangle, |\psi_{\text{Bell}_3}\rangle, |\psi_{\text{Bell}_4}\rangle\}, \quad (49)$$

where from (48) we get,

$$|\psi_{\text{Bell}_1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |\psi_{\text{Bell}_2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, |\psi_{\text{Bell}_3}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, |\psi_{\text{Bell}_4}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \quad (50)$$

From (16) and (48), it follows that the GA version of Bell states is given by,

$$\begin{aligned} |\psi_{\text{Bell}_1}\rangle &\leftrightarrow \psi_{\text{Bell}_1}^{(\text{GA})} = \frac{1}{2^{\frac{3}{2}}} (1 + i\sigma_2^1 i\sigma_2^2) (1 - i\sigma_3^1 i\sigma_3^2), |\psi_{\text{Bell}_2}\rangle \leftrightarrow \psi_{\text{Bell}_2}^{(\text{GA})} = -\frac{1}{2^{\frac{3}{2}}} (i\sigma_2^1 + i\sigma_2^2) (1 - i\sigma_3^1 i\sigma_3^2), \\ |\psi_{\text{Bell}_3}\rangle &\leftrightarrow \psi_{\text{Bell}_3}^{(\text{GA})} = \frac{1}{2^{\frac{3}{2}}} (1 - i\sigma_2^1 i\sigma_2^2) (1 - i\sigma_3^1 i\sigma_3^2), |\psi_{\text{Bell}_4}\rangle \leftrightarrow \psi_{\text{Bell}_4}^{(\text{GA})} = \frac{1}{2^{\frac{3}{2}}} (i\sigma_2^1 - i\sigma_2^2) (1 - i\sigma_3^1 i\sigma_3^2). \end{aligned} \quad (51)$$

We point out that within the MSTA formalism, there is no need either for an abstract spin space (the complex Hilbert space \mathcal{H}_2^n), containing objects which have to be operated on by quantum unitary operators (for instance, in the Bell states example, such operators are the CNOT gates), or for an abstract index convention. The requirement for an explicit matrix representation is also avoided. Within the MSTA formalism, the role of operators is taken over by

right or left multiplication by elements from the same geometric algebra as spinors (qubits) are taken from. This is an additional manifestation of a conceptual unification provided by GA - "spin (or qubit) space" and "unitary operators upon spin space" become united, with both being just multivectors in real space. Indeed, such a manifestation is not a special feature of our work since it appears in most geometric algebra applications to classical and quantum mathematical physics.

CNOT Quantum Gate. An convenient way to describe the CNOT quantum gate is the following [18],

$$\hat{U}_{\text{CNOT}}^{12} = \frac{1}{2} \left[\left(\hat{I}^1 + \hat{\Sigma}_3^1 \right) \otimes \hat{I}^2 + \left(\hat{I}^1 - \hat{\Sigma}_3^1 \right) \otimes \hat{\Sigma}_1^2 \right], \quad (52)$$

where $\hat{U}_{\text{CNOT}}^{12}$ is the CNOT gate from qubit 1 to qubit 2. It then follows that,

$$\hat{U}_{\text{CNOT}}^{12} |\psi\rangle = \frac{1}{2} \left(\hat{I}^1 \otimes \hat{I}^2 + \hat{\Sigma}_3^1 \otimes \hat{I}^2 + \hat{I}^1 \otimes \hat{\Sigma}_1^2 - \hat{\Sigma}_3^1 \otimes \hat{\Sigma}_1^2 \right) |\psi\rangle. \quad (53)$$

From (21) and (53), we obtain

$$\hat{I}^1 \otimes \hat{I}^2 |\psi\rangle \leftrightarrow \psi, \quad \hat{\Sigma}_3^1 \otimes \hat{I}^2 |\psi\rangle \leftrightarrow -i\sigma_3^1 \psi J, \quad \hat{I}^1 \otimes \hat{\Sigma}_1^2 |\psi\rangle \leftrightarrow -i\sigma_1^2 \psi J, \quad -\hat{\Sigma}_3^1 \otimes \hat{\Sigma}_1^2 |\psi\rangle \leftrightarrow i\sigma_3^1 i\sigma_1^2 \psi E. \quad (54)$$

Finally, from (53) and (54), we get the GA version of the CNOT gate,

$$\hat{U}_{\text{CNOT}}^{12} |\psi\rangle \leftrightarrow \frac{1}{2} \left(\psi - i\sigma_3^1 \psi J - i\sigma_1^2 \psi J + i\sigma_3^1 i\sigma_1^2 \psi E \right). \quad (55)$$

Controlled-Phase Gate. The action of \hat{U}_{CP}^{12} on $|\psi\rangle \in \mathcal{H}_2^2$ is given by [18],

$$\hat{U}_{\text{CP}}^{12} |\psi\rangle = \frac{1}{2} \left[\hat{I}^1 \otimes \hat{I}^2 + \hat{\Sigma}_3^1 \otimes \hat{I}^2 + \hat{I}^1 \otimes \hat{\Sigma}_3^2 - \hat{\Sigma}_3^1 \otimes \hat{\Sigma}_3^2 \right] |\psi\rangle. \quad (56)$$

From (21) and (56), we obtain

$$\hat{I}^1 \otimes \hat{I}^2 |\psi\rangle \leftrightarrow \psi, \quad \hat{\Sigma}_3^1 \otimes \hat{I}^2 |\psi\rangle \leftrightarrow -i\sigma_3^1 \psi J, \quad \hat{I}^1 \otimes \hat{\Sigma}_3^2 |\psi\rangle \leftrightarrow -i\sigma_3^2 \psi J, \quad -\hat{\Sigma}_3^1 \otimes \hat{\Sigma}_3^2 |\psi\rangle \leftrightarrow i\sigma_3^1 i\sigma_3^2 \psi E. \quad (57)$$

Finally, from (56) and (57), we obtain the GA version of the controlled-phase quantum gate,

$$\hat{U}_{\text{CP}}^{12} |\psi\rangle \leftrightarrow \frac{1}{2} \left(\psi - i\sigma_3^1 \psi J - i\sigma_3^2 \psi J + i\sigma_3^1 i\sigma_3^2 \psi E \right). \quad (58)$$

SWAP Gate. The action of $\hat{U}_{\text{SWAP}}^{12}$ on $|\psi\rangle \in \mathcal{H}_2^2$ is given by [18],

$$\hat{U}_{\text{SWAP}}^{12} |\psi\rangle = \frac{1}{2} \left(\hat{I}^1 \otimes \hat{I}^2 + \hat{\Sigma}_1^1 \otimes \hat{\Sigma}_1^2 + \hat{\Sigma}_2^1 \otimes \hat{\Sigma}_2^2 + \hat{\Sigma}_3^1 \otimes \hat{\Sigma}_3^2 \right) |\psi\rangle. \quad (59)$$

From (21) and (59), we obtain

$$\hat{I}^1 \otimes \hat{I}^2 |\psi\rangle \leftrightarrow \psi, \quad \hat{\Sigma}_1^1 \otimes \hat{\Sigma}_1^2 |\psi\rangle \leftrightarrow -i\sigma_1^1 i\sigma_1^2 \psi E, \quad \hat{\Sigma}_2^1 \otimes \hat{\Sigma}_2^2 |\psi\rangle \leftrightarrow -i\sigma_2^1 i\sigma_2^2 \psi E, \quad \hat{\Sigma}_3^1 \otimes \hat{\Sigma}_3^2 |\psi\rangle \leftrightarrow -i\sigma_3^1 i\sigma_3^2 \psi E. \quad (60)$$

Finally, from (59) and (60), we obtain the GA version of the SWAP gate,

$$\hat{U}_{\text{SWAP}}^{12} |\psi\rangle \leftrightarrow \frac{1}{2} \left(\psi - i\sigma_1^1 i\sigma_1^2 \psi E - i\sigma_2^1 i\sigma_2^2 \psi E - i\sigma_3^1 i\sigma_3^2 \psi E \right). \quad (61)$$

In conclusion, the action of some of the most relevant 2-qubit quantum gates in the GA formalism on the GA computational basis $\mathcal{B}_{[\text{cl}^+(3) \otimes \text{cl}^+(3)]/E}$ can be summarized in the following tabular form:

2-Qubit Gates	2-Qubit States	GA Action of Gates on States
CNOT	ψ	$\frac{1}{2} (\psi - i\sigma_3^1 \psi J - i\sigma_1^2 \psi J + i\sigma_3^1 i\sigma_1^2 \psi E)$
Controlled-Phase Gate	ψ	$\frac{1}{2} (\psi - i\sigma_3^1 \psi J - i\sigma_3^2 \psi J + i\sigma_3^1 i\sigma_3^2 \psi E)$
SWAP	ψ	$\frac{1}{2} (\psi - i\sigma_1^1 i\sigma_1^2 \psi E - i\sigma_2^1 i\sigma_2^2 \psi E - i\sigma_3^1 i\sigma_3^2 \psi E)$

(62)

Two-qubit quantum gates have a geometric interpretation in terms of rotations as well. For instance, the CNOT gate describes a rotation in one qubit space *conditional* on the state of another qubit it is correlated with. The general expression of the corresponding operator in GA is given by $\left(\hat{U}_{\text{CNOT}}^{12} \right)^{(\text{GA})} = e^{-i\frac{\pi}{2} \frac{1}{2} \sigma_1^1 (1 - \sigma_3^2)}$. This operator rotates the first qubit by π about the axis σ_1^1 in those 2-qubit states where the second qubit is along the $-\sigma_3^2$ axis. Similar geometric considerations could be considered for the other 2-qubit gates [10].

C. Geometric Algebra and Density Operators

For the sake of completeness, we point out that statistical aspects of quantum systems cannot be described in terms of a single wavefunction. Instead, they can be properly handled in terms of density matrices. The density matrix for a pure state is given by,

$$\hat{\rho}_{\text{pure}} = |\psi\rangle \langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}. \quad (63)$$

The expectation value of any observable \hat{O} associated with the state $|\psi\rangle$ can be obtained from $\hat{\rho}_{\text{pure}}$ by writing $\langle\psi|\hat{O}|\psi\rangle = \text{Tr}(\hat{\rho}_{\text{pure}}\hat{O})$. The GA version of $\hat{\rho}_{\text{pure}}$ is,

$$\hat{\rho}_{\text{pure}} \rightarrow \rho_{\text{pure}}^{(\text{GA})} = \psi \frac{1}{2} (1 + \sigma_3) \psi^\dagger = \frac{1}{2} (1 + s), \quad (64)$$

where $s \stackrel{\text{def}}{=} \psi \sigma_3 \psi^\dagger$ is the spin vector [16]. From a geometric point of view, $\rho_{\text{pure}}^{(\text{GA})}$ is just the sum of a scalar and a vector. The density matrix for a mixed state $\hat{\rho}_{\text{mixed}}$ is the weighted sum of the density matrices for the pure states,

$$\hat{\rho}_{\text{mixed}} = \sum_{j=1}^n \hat{\rho}_j = \sum_{j=1}^n p_j |\psi_j\rangle \langle\psi_j|, \quad (65)$$

with $p_j \in \mathbb{R}$ for $j = 1, \dots, n$ and $p_1 + \dots + p_n = 1$. In the GA formalism, addition is well-defined and the geometric algebra version of $\hat{\rho}_{\text{mixed}}$ becomes the sum,

$$\hat{\rho}_{\text{mixed}} \rightarrow \rho_{\text{mixed}}^{(\text{GA})} = \frac{1}{2} \sum_{j=1}^n (p_j + p_j s_j) = \frac{1}{2} (1 + P), \quad (66)$$

where P is the ensemble-average polarization vector (average spin vector) with length $\|P\| \leq 1$. The magnitude of P measures the degree of alignment among the unit length polarization vectors of the individual numbers of the ensemble. We point out that $\rho_{\text{mixed}}^{(\text{GA})}$ is the geometric algebra expression of the density operator of an ensemble of identical and non-interacting qubits. More generally, we could also consider density operators of interacting multi-qubit systems. The MSTA version of the density matrix of n -interacting qubits reads,

$$\rho_{\text{multi-qubit}}^{(\text{GA})} = \overline{(\psi E_n) E_+ (\psi E_n)^\sim}, \quad (67)$$

where E_n is the n -particle correlator and $E_+ \stackrel{\text{def}}{=} E_+^1 E_+^2 \dots E_+^n$ is the geometric product of n -idempotents with $E_\pm^k = \frac{1 \pm \sigma_x^k}{2}$ and $k = 1, \dots, n$. The symbol tilde denotes the space-time reverse and the over-line denotes the ensemble-average. A more detailed application of the GA formalism to general density matrices appears in [9].

IV. ON THE UNIVERSALITY OF QUANTUM GATES AND GEOMETRIC ALGEBRA

In this Section, using the above mentioned explicit characterization and the GA description of the Lie algebras $SO(3)$ and $SU(2)$ based on the rotor group $Spin^+(3, 0)$ formalism, we reexamine Boykin's proof of universality of quantum gates. In the first part, we introduce the rotor group. In the second part, we introduce few universal sets of quantum gates. In the last part, we present our GA-based proof.

A. On $SO(3)$, $SU(2)$ and the Rotor Group in Geometric Algebra

Since Boykin's proof heavily relies on the properties of rotations in three-dimensional space and on the local isomorphism between $SO(3)$ and $SU(2)$, we briefly present the the GA description of such Lie groups via the rotor group $Spin^+(3, 0)$.

1. Remarks on $SO(3)$ and $SU(2)$

Two important groups in physics are the 3-dimensional Lie groups $SO(3)$ with Lie algebra $\mathfrak{so}(3)$ and $SU(2)$ with Lie algebra $\mathfrak{su}(2)$ [23]. The former is the group of rotations of three-dimensional space, i.e. the group of orthogonal transformations with determinant 1,

$$SO(3) \stackrel{\text{def}}{=} \{M \in GL(3, \mathbb{R}) : MM^t = M^t M = I_{3 \times 3}, \det M = 1\}, \quad (68)$$

where " t " denotes the transpose of a matrix and $GL(3, \mathbb{R})$ is the set of non-singular linear transformations in \mathbb{R}^3 which are represented by 3×3 non singular matrices with real entries. The latter is the group of all 2×2 unitary complex matrices with determinant equal to 1,

$$SU(2) \stackrel{\text{def}}{=} \{M \in GL(2, \mathbb{C}) : MM^\dagger = M^\dagger M = I_{2 \times 2}, \det M = 1\}, \quad (69)$$

where " \dagger " denotes the Hermitian conjugate and $GL(2, \mathbb{C})$ is the set of non-singular linear transformations in \mathbb{C}^2 which are represented by 2×2 non singular matrices with complex entries. The Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic, $\mathfrak{so}(3) \cong \mathfrak{su}(2)$. The Lie groups $SO(3)$ and $SU(2)$ are *locally* isomorphic, they are indistinguishable at the level of infinitesimal transformations. However, they differ at a global level, i. e. far from identity. This means that $SO(3)$ and $SU(2)$ are not isomorphic. In $SO(3)$ a rotation by 2π is the same as the identity. Instead, $SU(2)$ is periodic only under rotations by 4π . This means that an object that picks a minus sign under a rotation by 2π is an acceptable representation of $SU(2)$, while it is not an acceptable representation of $SO(3)$. Spin $\frac{1}{2}$ particles or qubits need to be rotated 720° in order to come back to the same state [24]. Topologically, $SU(2)$ is the 3-sphere \mathcal{S}^3 , $SU(2) \approx \mathcal{S}^3$. Instead, $SO(3)$ is topologically equivalent to the projective space \mathbb{RP}^3 where \mathbb{RP}^3 results from \mathcal{S}^3 by identifying pairs of antipodal points. This leads to conclude the actual isomorphism between groups is $SU(2)/\mathbb{Z}_2 \cong SO(3)$. In formal mathematical terms, there is a not faithful representation \varkappa of $SU(2)$ as a group of rotations of \mathbb{R}^3 ,

$$\varkappa : SU(2) \ni U_{SU(2)}(\vec{A}, \theta) \stackrel{\text{def}}{=} \exp\left(\frac{\vec{\Sigma}}{2i_{\mathbb{C}}} \cdot \vec{A}\theta\right) \mapsto R_{SO(3)}(\vec{A}, \theta) \stackrel{\text{def}}{=} \exp(\vec{E} \cdot \vec{A}\theta) \in SO(3), \quad (70)$$

for any vector $\vec{A} = (A_1, A_2, A_3)$. For the sake of mathematical correctness, we point out that the use of the dot-notation in (70) (and in the following equations (72), (83), (106)) is indeed an abuse of notation for the Euclidean inner product because \vec{A} is geometrically just a vector in \mathbb{R}^3 while $\vec{\Sigma}$ are operators (Pauli matrices) on a two-dimensional Hilbert space. The vector $\vec{E} = (E_1, E_2, E_3)$ forms a basis of infinitesimal generators of the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$,

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (71)$$

and they satisfy the following commutation relations, $[E_l, E_m] = \varepsilon_{lmk} E_k$. The infinitesimal generators of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ are $i_{\mathbb{C}} \vec{\Sigma} = (i_{\mathbb{C}} \Sigma_1, i_{\mathbb{C}} \Sigma_2, i_{\mathbb{C}} \Sigma_3)$ satisfying the commutation relations, $[\Sigma_l, \Sigma_m] = 2i_{\mathbb{C}} \varepsilon_{lmk} \Sigma_k$. This commutations relations are the same as for $SO(3)$ if one uses $\frac{\Sigma_l}{2i_{\mathbb{C}}}$ as the new basis for $\mathfrak{su}(2)$. The map \varkappa is exactly 2 : 1 and therefore to a rotation of \mathbb{R}^3 about an axis given by a unit vector \vec{A} through an angle θ radians one associates two 2×2 unitary matrices with determinant 1,

$$\exp\left(\frac{\vec{\Sigma}}{2i_{\mathbb{C}}} \cdot \vec{A}\theta\right), \exp\left(\frac{\vec{\Sigma}}{2i_{\mathbb{C}}} \cdot \vec{A}(\theta + 2\pi)\right). \quad (72)$$

In other words, $SO(3)$ not only has the usual representation by 3×3 matrices, it also has a double-valued representation by 2×2 matrices acting on \mathbb{C}^2 . The complex vectors $\begin{pmatrix} \psi^1 & \psi^2 \end{pmatrix}^t \in \mathbb{C}^2$ on which $SO(3)$ acts in this double-valued way are called spinors. In mathematical terms, $SU(2)$ furnishes naturally a spinor representation of the 2-fold cover of $SO(3)$. When $SU(2)$ is thought of as the 2-fold cover of $SO(3)$, it is called the spin group $Spin(3)$. It is remarkably powerful to represent three-dimensional rotations in terms of two-dimensional unitary transformations. In quantum information science, this is especially true in proving certain circuit identities, in the characterization of the general 1-qubit state and in the construction of the Hardy state [25]. Indeed, this representation plays also a key role in the proof of universality of quantum gates provided by Boykin et al. For instance, the surjective homeomorphism \varkappa is a powerful tool for investigating the product of two or more rotations. This is a consequence of the fact that the Pauli matrices satisfy very simple product rules, $\Sigma_l \Sigma_m = \delta_{lm} + i_{\mathbb{C}} \varepsilon_{lmk} \Sigma_k$. The infinitesimal generators $\{E_l\}$ with $l = 1, 2, 3$ of $\mathfrak{so}(3)$ do not satisfy such simple product relations. For instance, $E_1^2 = \text{diag}(0, -1, -1)$.

2. Remarks on the rotor group

One of the most powerful applications of geometric algebra is to rotations. Within GA, rotations are handled through the use of *rotors*. Rotors also provide a convenient framework for studying Lie groups and Lie algebras. Let us introduce few definitions. More details on the mathematical structure of Clifford algebras appears in [26]

Let $\mathcal{G}(p, q)$ denote the GA of a space of signature p, q with $p+q = n$ and let \mathcal{V} be the space of grade-1 multivectors. Then, the pin group (with respect to the geometric product) $Pin(p, q)$ is defined as,

$$Pin(p, q) \stackrel{\text{def}}{=} \{M \in \mathcal{G}(p, q) : MaM^{-1} \in \mathcal{V} \forall a \in \mathcal{V}, MM^\dagger = \pm 1\}, \quad (73)$$

where " \dagger " denotes the reversion operation in GA (for instance, $(a_1 a_2)^\dagger = a_2 a_1$). The elements of the pin group split into even-grade and odd-grade elements. The even-grade multivectors $\{S\}$ of the pin group form a subgroup called the spin group $Spin(p, q)$,

$$Spin(p, q) \stackrel{\text{def}}{=} \{S \in \mathcal{G}_+(p, q) : SaS^{-1} \in \mathcal{V} \forall a \in \mathcal{V}, SS^\dagger = \pm 1\}, \quad (74)$$

where $\mathcal{G}_+(p, q)$ denotes the even subalgebra of $\mathcal{G}(p, q)$. Finally, rotors are elements $\{R\}$ of the spin group satisfying the further constraint that $RR^\dagger = +1$. These elements define the so-called rotor group $Spin^+(p, q)$,

$$Spin^+(p, q) \stackrel{\text{def}}{=} \{R \in \mathcal{G}_+(p, q) : RaR^\dagger \in \mathcal{V} \forall a \in \mathcal{V}, RR^\dagger = +1\}. \quad (75)$$

For Euclidean spaces, $Spin(n, 0) = Spin^+(n, 0)$. Therefore, for such spaces, there is no distinction between the spin group and the rotor group.

In GA, the rotation of a vector a through θ in the plane generated by two unit vectors m and n is defined by the double-sided half-angle transformation law,

$$a \rightarrow a' \stackrel{\text{def}}{=} RaR^\dagger. \quad (76)$$

The rotor R is defined by,

$$R \stackrel{\text{def}}{=} nm = n \cdot m + n \wedge m \equiv \exp\left(-B\frac{\theta}{2}\right), \quad (77)$$

where the bivector B is such that,

$$B = \frac{m \wedge n}{\sin \frac{\theta}{2}}, \quad B^2 = -1. \quad (78)$$

Rotors provide a way of handling rotations that is unique to GA and this is a consequence of the definition of the geometric product. Notice that rotors are geometric products of two unit vectors and therefore they are mixed-grade objects. The rotor R on its own has no significance, which is to say that no meaning should be attached to the separate scalar and bivector terms. When R is written as the exponential of the bivector B (all rotors near the origin can be written as the exponential of a bivector and the exponential of a bivector always returns to a rotor), however, the bivector has a clear geometric significance, as does the vector formed from RaR^\dagger . This illustrates a central feature of GA, which is that both geometrically meaningful objects (vectors, planes, etc.) and the elements (operators) that act on them (in this case, rotors or bivectors) are contained in the same geometric Clifford algebra. Notice that R and $-R$ generate the same rotation, so there is a two-to-one map between rotors and rotations. Formally, the rotor group provides a double-cover representation of the rotation group.

The Lie algebra of the rotor group $Spin^+(3, 0)$ is defined in terms of the bivector algebra,

$$[B_l, B_m] = 2B_l \times B_m = -2\varepsilon_{lmk}B_k, \quad (79)$$

where " \times " denotes the commutator product of two multivectors in GA and,

$$B_1 = \sigma_2 \sigma_3 = i\sigma_1, \quad B_2 = \sigma_3 \sigma_1 = i\sigma_2, \quad B_3 = \sigma_1 \sigma_2 = i\sigma_3. \quad (80)$$

The commutator of a bivector with a second bivector produces a third bivector. That is, the space of bivectors is closed under the commutator product. This closed algebra defines the Lie algebra of the associated rotor group. The group is formed by the act of exponentiation. Furthermore, notice that the product of bivectors satisfies,

$$B_l B_m = -\delta_{lm} - \varepsilon_{lmk} B_k. \quad (81)$$

The symmetric part of this product is a scalar, whereas the antisymmetric part is a bivector. As a final remark, we point out that the algebra of bivectors is similar to the algebra of the generators of the quaternions. Thus, quaternions can be identified with bivectors within the GA approach. The relations among $SO(3)$, $SU(2)$ and the rotor group are summarized as follows,

<i>Lie Groups</i>	<i>Lie Algebras</i>	<i>Product Rules</i>	<i>Operator, Vectors</i>
$SO(3)$	$[E_l, E_m] = \varepsilon_{lmk} E_k$	not useful	Orthogonal transformations, vectors in \mathbb{R}^3
$SU(2)$	$[\Sigma_l, \Sigma_m] = 2i_{\mathbb{C}} \varepsilon_{lmk} \Sigma_k$	$\Sigma_l \Sigma_m = \delta_{lm} + i_{\mathbb{C}} \varepsilon_{lmk} \Sigma_k$	Unitary operators, spinors
$Spin^+(3, 0)$	$[B_l, B_m] = -2\varepsilon_{lmk} B_k$	$B_l B_m = -\delta_{lm} - \varepsilon_{lmk} B_k$	Rotors (or, bivectors), multivectors

(82)

Therefore, two central features of GA emerge: 1) the GA provides a very clear and compact method for encoding rotations which is considerably more powerful than working with matrices; 2) Both geometrically meaningful objects (vectors, planes, etc.) and the elements (operators) that act on them (in this case, rotors or bivectors) are contained in the same geometric Clifford algebra.

B. Universal Sets of Quantum Gates

Quantum computational gates are input-output devices whose inputs and outputs are discrete quantum variables such as spins. As a matter of fact, recall that the most general 2×2 unitary matrix with determinant 1 can be expressed in the form of a finite rotation represented as,

$$\hat{U}_{SU(2)}(\hat{n}, \theta) \stackrel{\text{def}}{=} e^{-i_{\mathbb{C}} \frac{\theta}{2} \hat{n} \cdot \vec{\Sigma}} = \hat{I} \cos\left(\frac{\theta}{2}\right) - i_{\mathbb{C}} \hat{n} \cdot \vec{\Sigma} \sin\left(\frac{\theta}{2}\right). \quad (83)$$

Therefore, we are entitled to think of a qubit as the state of a spin- $\frac{1}{2}$ object and an arbitrary unitary transformation (quantum gate) acting on the state (aside from a possible rotation of the overall phase) is a rotation of the spin. A set of gates is *adequate* if any quantum computation can be performed with arbitrary precision by networks consisting only of replicas of gates from that set. A gate is *universal* if by itself it forms an adequate set, i. e. if any quantum computation can be performed by a network containing replicas of only this gate. The first example of universal gate is the universal Deutsch three-bit gate [27]. In the network's computational basis $\mathcal{B}_{\mathcal{H}_2^3} = \{|000\rangle, |100\rangle, |010\rangle, |001\rangle, |110\rangle, |101\rangle, |011\rangle, |111\rangle\}$, it is given by the 8×8 unitary matrix $\mathcal{D}_{\text{universal}}^{(\text{Deutsch})}(\gamma)$,

$$\mathcal{D}_{\text{universal}}^{(\text{Deutsch})}(\gamma) \stackrel{\text{def}}{=} \begin{pmatrix} I_{6 \times 6} & O_{6 \times 2} \\ O_{2 \times 6} & D_{2 \times 2}(\gamma) \end{pmatrix}, \quad (84)$$

where $I_{l \times k}$ is the $l \times k$ identity matrix, $O_{l \times k}$ is the $l \times k$ null matrix and $D_{2 \times 2}(\gamma)$ is defined as,

$$D_{2 \times 2}(\gamma) \stackrel{\text{def}}{=} \begin{pmatrix} i_{\mathbb{C}} \cos\left(\frac{\pi\gamma}{2}\right) & \sin\left(\frac{\pi\gamma}{2}\right) \\ \sin\left(\frac{\pi\gamma}{2}\right) & i_{\mathbb{C}} \cos\left(\frac{\pi\gamma}{2}\right) \end{pmatrix}. \quad (85)$$

The Deutsch gate depends on the parameter γ that can be any irrational number. Another important example of universal quantum gate is given by the Barenco three-parameter family of universal two-bit gates [28]. In the network's computational basis $\mathcal{B}_{\mathcal{H}_2^2} = \{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, it is given by the 4×4 unitary matrix $\mathcal{A}_{\text{universal}}^{(\text{Barenco})}(\phi, \alpha, \theta)$,

$$\mathcal{A}_{\text{universal}}^{(\text{Barenco})}(\phi, \alpha, \theta) \stackrel{\text{def}}{=} \begin{pmatrix} I_{2 \times 2} & O_{2 \times 2} \\ O_{2 \times 2} & A_{2 \times 2}(\phi, \alpha, \theta) \end{pmatrix}, \quad (86)$$

where $I_{l \times k}$ is the $l \times k$ identity matrix, $O_{l \times k}$ is the $l \times k$ null matrix and $A_{2 \times 2}(\phi, \alpha, \theta)$ is defined as,

$$A_{2 \times 2}(\phi, \alpha, \theta) \stackrel{\text{def}}{=} \begin{pmatrix} e^{i_{\mathbb{C}} \alpha} \cos \theta & -i_{\mathbb{C}} e^{i_{\mathbb{C}}(\alpha - \phi)} \sin \theta \\ -i_{\mathbb{C}} e^{i_{\mathbb{C}}(\alpha + \phi)} \sin \theta & e^{i_{\mathbb{C}} \alpha} \cos \theta \end{pmatrix}. \quad (87)$$

The Barenco gate depends on three parameters ϕ , α , and θ that are fixed irrational multiples of π and of each other. More generally, it turns out that *almost* all two-bit (and more inputs) quantum gates are universal [29, 30].

A set of quantum gates \mathcal{S} is said to be universal if an arbitrary unitary quantum operation can be performed with arbitrarily small error probability using a quantum circuit that only uses gates from \mathcal{S} . An important set of logic gates in quantum computing is given by,

$$\mathcal{S}_{\text{Clifford}} \stackrel{\text{def}}{=} \left\{ \hat{H}, \hat{P}, \hat{U}_{\text{CNOT}} \right\}. \quad (88)$$

The set $\mathcal{S}_{\text{Clifford}}$ of Hadamard- \hat{H} , phase- \hat{P} and CNOT- \hat{U}_{CNOT} gates generates the so-called Clifford group, the normalizer $\mathcal{N}(\mathcal{G}_n)$ of the Pauli group \mathcal{G}_n in $\mathcal{U}(n)$ [31]. This set of gates is sufficient to perform fault-tolerant quantum computation but $\mathcal{S}_{\text{Clifford}}$ is not sufficiently powerful to perform universal quantum computation. However, universal quantum computation becomes possible if the gates in the Clifford group are supplemented with the Toffoli gate [32],

$$\mathcal{S}_{\text{universal}}^{(\text{Shor})} \stackrel{\text{def}}{=} \left\{ \hat{H}, \hat{P}, \hat{U}_{\text{CNOT}}, \hat{U}_{\text{Toffoli}} \right\}. \quad (89)$$

Shor showed that adding the Toffoli gate to the generators of the Clifford group produces the universal set $\mathcal{S}_{\text{universal}}^{(\text{Shor})}$. Another example of universal set of logic gates is provided by Boykin et al. in [19, 20]. The set they construct is given by,

$$\mathcal{S}_{\text{universal}}^{(\text{Boykin et al.})} \stackrel{\text{def}}{=} \left\{ \hat{H}, \hat{P}, \hat{T}, \hat{U}_{\text{CNOT}} \right\}. \quad (90)$$

This set is presumably easier to implement experimentally than $\mathcal{S}_{\text{universal}}^{(\text{Shor})}$ since the \hat{T} is a one-qubit gate while the Toffoli gate is a three-qubit gate.

C. GA reexamination of Boykin's proof of universality

Boykin's proof is very elegant and is solely based on the geometry of real rotations in three dimensions and on the local isomorphism between the Lie groups $SO(3)$ and $SU(2)$. In what follows, we will revisit the proof using a GA approach based on the rotor group $Spin^+(3, 0)$ and on the algebra of bivectors, $[B_l, B_m] = -2\epsilon_{lmk}B_k$.

The proof of universality of the basis $\mathcal{S}_{\text{universal}}^{(\text{Boykin et al.})}$ can be presented in two steps. In the first step, it is required to show that Hadamard gate \hat{H} and the $\frac{\pi}{8}$ -phase gate $\hat{T} = \hat{\Sigma}_3^{\frac{1}{4}}$ form a *dense* set in $SU(2)$ where,

$$\hat{\Sigma}_3^\alpha \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\mathbb{C}\pi\alpha} \end{pmatrix}, \quad \hat{\Sigma}_3^\alpha |\psi\rangle \leftrightarrow \psi_{\hat{\Sigma}_3^\alpha}^{(\text{GA})} = \sigma_3^\alpha \psi \sigma_3. \quad (91)$$

This means that any element $\hat{U}_{SU(2)}$ in $SU(2)$ can be approximated to a desired degree of precision by a finite product of \hat{H} and \hat{T} . In other words, when a circuit of quantum gates is used to implement some desired unitary operation \hat{U} , it is sufficient to have an implementation that approximates \hat{U} to some specified level of accuracy. Suppose we approximate \hat{U} by some other unitary transformation \hat{U}' . Then, the notion of the quality of an approximation of a unitary transformation can be quantified considering the so-called approximation error $\varepsilon(\hat{U}, \hat{U}')$ [33],

$$\varepsilon(\hat{U}, \hat{U}') \stackrel{\text{def}}{=} \max_{|\psi\rangle} \left\| (\hat{U} - \hat{U}') |\psi\rangle \right\|, \quad (92)$$

where $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ is the Euclidean norm of $|\psi\rangle$ and $\langle\cdot|\cdot\rangle$ is the conventional inner product defined on the complex Hilbert space. In the second step of the proof, it is necessary to point out that for universal computation all that is needed is \hat{U}_{CNOT} and $SU(2)$ [34].

To show that \hat{H} and \hat{T} form a *dense* set in $SU(2)$, the local isomorphism between $SO(3)$ and $SU(2)$ must be exploited. Indeed, it can be shown that using the set $\left\{ \hat{H}, \hat{T} = \hat{\Sigma}_3^{\frac{1}{4}} \right\}$, we can construct quantities in this basis that correspond to rotations by angles that are irrational multiples of π in $SO(3, \mathbb{R})$ about two orthogonal axes. Consider the following two rotations in $SO(3)$ described in terms of rotors in $Spin^+(3, 0)$,

$$SO(3) \ni \hat{U}_{SO(3)}^{(1)} \stackrel{\text{def}}{=} e^{i\mathbb{C}\lambda_1\pi\hat{n}_1\cdot\vec{\Sigma}} \leftrightarrow e^{in_1\lambda_1\pi} \in Spin^+(3, 0), \quad \hat{U}_{SO(3)}^{(2)} \stackrel{\text{def}}{=} e^{i\mathbb{C}\lambda_2\pi\hat{n}_2\cdot\vec{\Sigma}} \leftrightarrow e^{in_2\lambda_2\pi}, \quad (93)$$

where the elements λ_1, λ_2 are irrational numbers in \mathbb{R}/\mathbb{Q} . Let us show that rotations in (93) can be expressed in terms of a suitable combination of elements in $\left\{ \hat{H}, \hat{T} = \hat{\Sigma}_3^{\frac{1}{4}} \right\}$. It turns out that since $SU(2)/\mathbb{Z}_2 \cong SO(3)$, we have

$$Spin^+(3, 0) \ni e^{in_1\lambda_1\pi} \leftrightarrow \hat{U}_{SU(2)}^{(1)} \stackrel{\text{def}}{=} \hat{\Sigma}_3^{-\frac{1}{4}} \hat{\Sigma}_1^{\frac{1}{4}} \in SU(2) \text{ and } e^{in_2\lambda_2\pi} \leftrightarrow \hat{U}_{SU(2)}^{(2)} \stackrel{\text{def}}{=} \hat{H}^{-\frac{1}{2}} \hat{\Sigma}_3^{-\frac{1}{4}} \hat{\Sigma}_1^{\frac{1}{4}} \hat{H}^{\frac{1}{2}}, \quad (94)$$

where $\hat{\Sigma}_1^{\frac{1}{4}} = \hat{H}\hat{\Sigma}_3^{\frac{1}{4}}\hat{H}$. Working out the details of [19, 20] and using the results presented in Section III, it turns out that the rotor representation of $\hat{U}_{SU(2)}^{(1)}$ and $\hat{U}_{SU(2)}^{(2)}$ is given by,

$$\hat{U}_{SU(2)}^{(1)} \leftrightarrow R_1 = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} i\sigma_1 + \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right) i\sigma_2 + \frac{1}{2\sqrt{2}} i\sigma_3, \quad (95)$$

and,

$$\hat{U}_{SU(2)}^{(2)} \leftrightarrow R_2 = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) - \frac{1}{2} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) i\sigma_1 + \frac{1}{2} i\sigma_2 + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) i\sigma_3, \quad (96)$$

respectively. R_1 and R_2 are rotors in $Spin^+(3, 0)$. Notice that,

$$e^{in_k \lambda_k \pi} = \cos(\lambda_k \pi) + n_{kx} \sin(\lambda_k \pi) i\sigma_1 + n_{ky} \sin(\lambda_k \pi) i\sigma_2 + n_{kz} \sin(\lambda_k \pi) i\sigma_3, \quad (97)$$

for $k = 1, 2$ and unit vectors n_k . Therefore, setting $e^{in_1 \lambda_1 \pi} = R_1$ we get,

$$\cos(\lambda_1 \pi) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right), \quad n_{1y} \sin(\lambda_1 \pi) = \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right), \quad n_{1z} \sin(\lambda_1 \pi) = \frac{1}{2} \frac{1}{\sqrt{2}}, \quad n_{1x} = -n_{1z}. \quad (98)$$

Finally, after some algebra, we obtain that λ_1 is equal to,

$$\lambda_1 = \frac{1}{\pi} \cos^{-1} \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) \right], \quad (99)$$

and the unit vector $n_1 = n_{1x}\sigma_1 + n_{1y}\sigma_2 + n_{1z}\sigma_3$ is such that,

$$(n_{1x}, n_{1y}, n_{1z}) = \frac{1}{\sqrt{1 - \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) \right]^2}} \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right), \frac{1}{2\sqrt{2}} \right). \quad (100)$$

Similarly, setting $e^{in_2 \lambda_2 \pi} = R_2$, we obtain,

$$\cos(\lambda_2 \pi) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right), \quad n_{2y} \sin(\lambda_2 \pi) = \frac{1}{2}, \quad n_{2z} \sin(\lambda_2 \pi) = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right), \quad n_{2x} = -n_{2z}. \quad (101)$$

Finally, after some algebra, we obtain that $\lambda_2 = \lambda_1$ is equal to,

$$\lambda_2 = \frac{1}{\pi} \cos^{-1} \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) \right], \quad (102)$$

and the unit vector $n_2 = n_{2x}\sigma_1 + n_{2y}\sigma_2 + n_{2z}\sigma_3$ is such that,

$$(n_{2x}, n_{2y}, n_{2z}) = \frac{1}{\sqrt{1 - \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) \right]^2}} \left(-\frac{1}{2} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right), \frac{1}{2}, \frac{1}{2} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) \right). \quad (103)$$

From (100) and (103), it follows that $n_1 \cdot n_2 = 0$. Since $\lambda_1 = \lambda_2 \equiv \lambda \in \mathbb{R}/\mathbb{Q}$, there exist some $n \in \mathbb{N}$ such that any phase factor $e^{ic\phi}$ can be approximated by $e^{icn\lambda\pi}$,

$$e^{ic\phi} \approx e^{icn\lambda\pi}, \quad n \in \mathbb{N}. \quad (104)$$

From (94) and (104), it follows that we have at least two dense subsets of $SU(2, \mathbb{C})$, that is to say $e^{in_1\alpha}$ and $e^{in_2\beta}$ with,

$$\alpha \approx \lambda\pi l \pmod{2\pi} \text{ and } \beta \approx \lambda\pi l \pmod{2\pi} \text{ with } l \in \mathbb{N}. \quad (105)$$

Since n_1 and n_2 are orthogonal vectors, we can write any element $\hat{U}_{SU(2)} \in SU(2, \mathbb{C})$ in the following form,

$$\hat{U}_{SU(2)} = e^{ic\phi\hat{n}\cdot\vec{\Sigma}} \leftrightarrow e^{in\phi} = e^{in_1\alpha} e^{in_2\beta} e^{in_1\gamma}. \quad (106)$$

Notice that the representation in (106) is analogous to Euler rotations about three orthogonal vectors. Expansion of the LHS of (106) leads to,

$$e^{in\phi} = \cos \phi + in \sin \phi. \quad (107)$$

Expansion of the RHS of (106) yields,

$$e^{in_1\alpha} e^{in_2\beta} e^{in_1\gamma} = (\cos \alpha + in_1 \sin \alpha) (\cos \beta + in_2 \sin \beta) (\cos \gamma + in_1 \sin \gamma). \quad (108)$$

Recalling that $n_1 n_2 = n_1 \cdot n_2 + n_1 \wedge n_2$ and that the unit vectors n_1 and n_2 are orthogonal, we have,

$$n_1 n_2 = -n_2 n_1. \quad (109)$$

Moreover, recalling that,

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta, \text{ and, } \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, \quad (110)$$

further expansion of (108) together with the use of (109) and (110), leads to

$$e^{in\phi} = \cos \beta \cos(\alpha + \gamma) + \cos \beta \sin(\alpha + \gamma) in_1 + \sin \beta \cos(\gamma - \alpha) in_2 + \sin \beta \sin(\gamma - \alpha) n_1 \wedge n_2. \quad (111)$$

Setting (107) equal to (111), we finally obtain

$$\cos \phi = \cos \beta \cos(\alpha + \gamma) \quad (112)$$

and,

$$n \sin \phi = \cos \beta \sin(\alpha + \gamma) n_1 + \sin \beta \cos(\gamma - \alpha) n_2 - i \sin \beta \sin(\gamma - \alpha) (n_1 \wedge n_2). \quad (113)$$

In conclusion, the parameters α , β and γ can be found by inverting (112) and (113) for any element in $SU(2)$. Then, using the fact that \hat{U}_{CNOT} and $SU(2)$ form a universal basis for quantum computing [34], the proof is completed [19, 20]. It is evident that the GA provides a very clear and compact method for encoding rotations which is considerably more powerful than working with matrices. Furthermore, from a conceptual point of view, a central feature of GA emerges as well (although such feature appears in most geometric algebra applications): both vectors (grade-1 multivectors), planes (grade-2 multivectors) and the operators acting on them (in this case, rotors R or bivectors B) are contained in the same geometric Clifford algebra.

V. CONCLUSIONS AND REMARKS

In this article, we investigated the utility of GA methods in two specific applications to quantum information science. First, we presented an explicit multiparticle spacetime algebra description of one and two-qubit quantum states together with a MSTA characterization of one and two-qubit quantum computational gates. Second, using the above mentioned explicit characterization and the GA description of the Lie algebras $SO(3)$ and $SU(2)$ based on the rotor group $Spin^+(3, 0)$ formalism, we reexamined Boykin's proof of universality of quantum gates. We conclude that the MSTA approach leads to a useful conceptual unification where the complex qubit space and the complex space of unitary operators acting on them become united, with both being made just by multivectors in real space [35]. Furthermore, the GA approach to rotations based on the rotor group clearly brings conceptual and computational advantages compared to standard vectorial and matricial approaches. In what follows, we present few concluding remarks.

In standard quantum computation, the basic operation is the tensor product " \otimes ". In the GA approach to quantum computing, the basic operation becomes the geometric (Clifford) product. Tensor product has no neat geometric visualization while geometric product has clear geometric interpretations. For instance, it forms a cube ($\sigma_1 \sigma_2 \sigma_3$) from a vector (σ_1) and a square ($\sigma_2 \sigma_3$), an oriented square ($\sigma_1 \sigma_2$) from two vectors (σ_1 and σ_2), a square ($\sigma_2 \sigma_3$) from a cube ($\sigma_1 \sigma_2 \sigma_3$) and a vector (σ_1), and so on. Furthermore, entangled quantum states are replaced by multivectors with a clear geometric interpretation. For instance, a general multivector M in $\mathcal{Cl}(3)$ is a linear combination of blades, geometric products of different basis vectors supplemented by the identity 1 (basic oriented scalar),

$$M \stackrel{\text{def}}{=} M_0 1 + \sum_{j=1}^3 M_j \sigma_j + \sum_{j < k} M_{jk} \sigma_j \sigma_k + M_{123} \sigma_1 \sigma_2 \sigma_3, \text{ with } j, k = 1, 2, 3. \quad (114)$$

Entangled states are replaced by GA multivectors that are nothing but bags of shapes (points, 1; lines, σ_j ; squares, $\sigma_j\sigma_k$; cubes, $\sigma_1\sigma_2\sigma_3$; and so on).

As a final remark, we point out that one of the most fascinating open problems in quantum physics is the characterization of the complexity of quantum motion. In quantum information science, the concept of complexity is also defined for quantum unitary operators, the so-called quantum gate complexity (a quantitative measure for the computational work needed to accomplish a given task [18]). We believe that the conceptual unification between spaces of quantum states and of quantum unitary operators acting on such states provided by multiparticle geometric algebras may allow also for the possibility of providing a single mathematical framework where complexities of both quantum states and quantum gates are defined for quantities both belonging to the same *real* multivectorial space (the reality of the multivectorial space is required for geometric purposes). Furthermore, this unification may turn out to be very useful in view of the recent connection made between quantum gate complexity and complexity of the motion on a suitable Riemannian manifold of multi-qubit unitary transformations provided by Nielsen and coworkers [36, 37].

In view of such considerations, we believe that the application of geometric Clifford algebras to the characterization of quantum gate complexity and to quantum information science in general is worthy of further investigations.

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